Term Structure Models: Estimation and Theory

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1 Introduction

Term structures model the relationship between time to maturity and yields for bonds, usually zero-coupon Treasury bonds, and the resulting curve is known as the yield curve. We develop models for estimating the US yield curve using various parametric methods, specifically the cubic B-spline, smoothing spline, and exponential polynomial methods. Then we discuss the theoretical framework of term structure models, including Hull-White and the one and two factor CIR models.

2 Yield Curve Estimation

There are two general classes of models the term structure models of interest rates: *parametric* methods and *nonparametric* methods. While the nonparametric models offer flexibility that is often desirable for real-world applications, the parametric models are easier to interpret and also have nice mathematical properties. This paper specifically focuses on the cubic spline, smoothing spline, and Svensson model (a type of exponential polynomial model).

Splines in a general sense refer to continuous piecewise polynomials . Using splines for data interpolation and smoothing is a popular method for modeling the yield curve. The cubic and smoothing splines that we examine in this paper use a set $b = b_0, b_1, ...b_n$ of control points (i.e. the price/yields of the Treasury bonds) to construct a smooth piecewise curve. Besides the benefits of close approximation of the yield curve, both cubic and smoothing splines are C^2. We begin with background on splines, introducing Bezier curves and Bezier splines before formally introducing cubic B-splines, smoothing splines, and the Svensson model.

2.1 Par Curves and Spot Curves

In any discussion of yield curves, it is necessary to explain the difference between the par yield curve and the spot yield curve. Coupon-paying bonds typically pay a fixed coupon semi-annually and then pay back the par value at maturity. The yield-to-maturity (YTM) is the single discount rate that can be used on the cash flows of a coupon-paying bond to determine its present value, or market price. The yield-to-maturity as a function of maturity is given by the par curve.

Whereas the par curve is used with coupon-paying bonds, or instruments with multiple cash flows, the spot curve is used with zero-coupon bonds. That is, the spot rate is used to discount a single cash flow at maturity to determine its present value. For this reason, the spot curve is also known as the discount function, since it provides the discount factor of a future cash flow for any maturity. To determine the present value of a coupon-paying bond using the spot curve, each individual cash flow of the bond can be treated as an individual zero-coupon bond and discounted at its maturity. This connection will be re-visited below in the application of B-splines.

2.2 Bezier Curves

A Bezier curve is a smooth parametric, polynomial curve constructed using and intended to fit a set of $n+1$ coefficients, called **control points**, and a set of basis polynomials, where n is the degree. Rather than the standard polynomial basis, Bezier curves use Bernstein polynomials as a basis. The Bernstein polynomials are given below for degree 3:

$$B_0(t) = t^3 \quad B_1(t) = 3t^2(1-t) \quad B_2(t) = 3t^(1-t)^2 \quad B_3(t) = (1-t)^3$$

Accordingly, given a set of 4 control points (p_0, p_1, p_2, p_3), the corresponding cubic Bezier curve on [a,b] is defined as:

$$C_{[a,b]}(t) = \sum_{i=0}^{3} p_i B_i(t) \quad a \leq t \leq b$$

Also, the Bezier curve must vanish outside of the interval [a,b]. When $t = 0$, all Bernstein polynomials equal 0 except for $B_1(a) = 1$, and when $t = 1$, all Bernstein polynomials equal 0 except $B_3(b) = 1$. Thus, we have that $C_{[a,b]}(a) = p_0$ and $C_{[a,b]}(b) = p_3$ so the Bezier curve will pass through the first and last control points at $t = a$ and $t = b$, respectively. A Bezier curve need not pass through any of the intermediate control points.

2.3 Bezier Splines

Definition 2.1. *Bezier Cubic Spline:*
A Bezier spline, is a piecewise function composed of Bezier curves.

Following from above, it is valuable to limit the degree of the Bezier curves making up the segments of a spline to prevent overfitting and complexity. A widely used implementation is the cubic Bezier spline, where each segment of the spline is a cubic Bezier curve. To build a cubic Bezier spline, $n+1$ provided control points are used to generate a new set of $n+3$ *de Boor control points*, which are then used as the standard control points for each individual curve segment. This new set of control points is used to generate cubic Bezier segments. These Bezier segments meet at junction points and typically follow smoothness criteria that ensure that the curve is twice differentiable. What is unique about Bezier splines is that the only input is the set of control points. That is, the de Boor control points and the junctions, known as *knots*, are not given as input but rather calculated using a system of equations. In other spline implementations, such as the B-spline, knots are provided as an input in the construction of the curve.

2.4 Cubic B-Spline

The B-spline, or the basis spline, is a modification of the Bezier spline that produces stabler results. A B-spline $C(t)$ is given by the equation:

$$C(t) = \sum_{i=0}^{n} b_i N_{i,k}(t) \quad 0 \leq t \leq 1$$

Whereas the Bezier Spline uses the set of Bernstein polynomials as its basis, the B-spline uses the set of $N_{i,k}$ functions, indexed by i and with degree k, as its basis. A B-spline has $n+1$ control points, $m+1$ knots, and basis functions with degree k. These parameters relate by the equation $m = n + k + 1$.

Cubic B-splines are most widely used in term structure modeling. Since basis functions have degree 3, the relationship between knots and control points is refined to $m = n + 4$.

Since the B-spline curve is a linear combination of the basis functions using control points, there are as many basis functions as control points. The basis functions can be calculated using a set of chosen knot points. That is, B-splines offer more flexibility than Bezier splines do, which improves correctness of fit when the knots are chosen wisely. Given $q + 7$ knots,

$$\xi_{-3} < \xi_{-2} < ... < \xi_q < \xi_{q+1} < \xi_{q+2} < \xi_{q+3}$$

the $q + 3$ basis functions are given by

$$N_{k,3} = \sum_{j=k}^{k+4} (\prod_{i=k,i\ neqj}^{k+4} (\frac{1}{\xi_i - \xi_j}))(x - \xi_j)^3_+ \qquad k = -3, -2, ..., q-1$$

2.5 Modeling Term Structures

To model the term structure, a cubic B-spline must be defined to model the discount function, or spot curve, which predicts the price of a bond as a percentage of its face value at a maturity time t. This spline, D, is parameterized by a vector of control points z. The present value of a zero-coupon bond, or any cash flow, that pays face value \$1 at time t is then given by a B-spline curve of the form:

$$D(t; z) = z_1 N_{1,3}(t) + z_2 N_{2,3}(t) + ... + z_{x-1} N_{x-1,3}(t) + z_x N_{x,3}(t)$$

This curve is described by $x + 1$ control points, $z = (z_0, z_1, ..., z_x)$, and corresponding basis functions. Further, we can describe a length-m column vector of discount factors that the cubic B-spline predicts at times $t_1, t_2, ..., t_m$. This vector is given by

$$d(z) = \begin{pmatrix} D(t_1; z) \\ D(t_2; z) \\ \vdots \\ D(t_{m-1}; z) \\ D(t_m; z) \end{pmatrix} = \begin{pmatrix} N_{0,3}(t_1) & \dots & N_{x,3}(t_1) \\ \vdots & & \vdots \\ N_{0,3}(t_{m-1}) & \dots & N_{x,3}(t_{m-1}) \\ N_{0,3}(t_m) & \dots & N_{x,3}(t_m) \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{x-1} \\ z_x \end{pmatrix} =: \Psi z$$

We note that each value $d(z)_i$ in the column vector, for $1 \leq i \leq m$, provides the spot rate at maturity t_i. Therefore, given a length-m row vector of cash flows, call it c, where each item c_i represents the cash flow at time t_i, the dot product $c \cdot d(z)$ gives the present value of the m cash flows. This follows from the description of par curves and spot curves above. That is, these m cash flows can be considered to be the cash flows of a coupon-paying bonds, and the dot product with the spot rate column vector $d(z)$ discounts each cash flow as an individual zero-coupon bond and sums the values.

As data input, we use a vector of n quoted market bond prices $p = (p_1, p_2, ..., p_n)^T$ for several coupon-paying bonds. Even though the objective is to model the term structure for zero-coupon bonds, most market data exists for coupon-paying bonds. Additionally, an n-by-m cash flow matrix, C, is needed to provide the coupon payment cash flows of these n bonds over m times, where $c_{i,j}$ represents the cash flow of bond p_i at time t_j. Then, in this application, the control points are calculated with the objective of minimizing the difference between the quoted bond prices and the predicted price using the cubic B-spline.

Using matrix C and discount factors $d(z)$, we can define the length-n column vector A by the dot product

$$A = C \cdot \Psi_z$$

where element A_i gives the prediction of the price of bond p_i. The task of determining the set of control points can then be given by the linear optimization

$$\min_{z \in \mathbb{R}^{x+1}} \left\| P - C \cdot \Psi_z \right\|^2$$

which calculates the control points that give the smallest difference between the quoted bond prices and the prices predicted by the cubic B-spline modeling the discount function. Assuming A has full rank, the optimal solution is given by the classical regression coefficient estimator:

$$z^* = (A^T A)^{-1} A^T p$$

2.6 Evaluation of B-Splines

For the above example, the cubic B-spline that models the discount function can be evaluated using the optimization function above. This measure is given by:

$$\min_{z \in \mathbb{R}^{x+1}} \left\| P - C \cdot \Psi_z \right\|^2 = \left\| P - C \cdot \Psi_{z^*} \right\|^2$$

A discussion of knots was ignored in the above example. Because there are $x + 1$ control points, and the degree is $k = 3$, we can determine the number of required knots $s + 1$ using the relation $s = x+1+k$, or $s+1 = x+5$. Selection of these $(x+5)$ knots is required to define the $x + 1$ B-spline basis functions. Above, it was noted that *wise* selection of the number of and location of knot point improves correctness of fit. Consequently, it is more typical that knots are chosen first as parameters, and so the number of control points depends on this selection. Both the number of and the selection of knot points have significant impact on the resulting B-spline curve.

When more knot points are used, more control points are required. The resulting cubic B-spline has a better fit although the regularity is reduced. With fewer knot points, fewer control points are required. The resulting cubic B-spline has a worse fit but the curve is more regular. There is clearly a trade-off between regularity and correctness of fit.

When modeling term-structures, B-splines perform worst for very short-term and long-term maturities, especially because manual selection of knots is challenging. An adjacent concept, smoothing splines refine the selection of the location and number of the knots so that curve regularity and correctness of fit are optimized.

2.7 Smoothing Splines

Smoothing Splines allow us to weight the objectives of goodness of fit with smoothness. In contrast to the cubic B-spline, the smoothing spline relies on the forward curve to estimate the yield curve. For simplicity, the rest of the models presented in this paper are for zero-coupon bonds (i.e. $C = I_n$). Then the forward rate $f(t,T)$ is the instantaneous rate that can be locked in now for borrowing at time T for $T \geq t$, and the forward curve is then the function $T \to f(t,T)$. For the sake of convenience, we will set $t = 0$ (no time like the present!) and let $f(0,u) = f(u)$. After estimating the optimal forward curve, we will work backwards to get zero-coupon bond prices and yields.

Like the cubic-Bspline, we can formulate the smoothing spline as a minimization problem. However instead of $\min\limits_{z\in\mathbb{R}^m} \left\|P - Cd(z)\right\|^2$, we have:

$$\min_f F(f) \text{ where F is a nonlinear functional}$$

$$F(f) = \int_0^T (f'(u))^2 du + \alpha \sum_{i=1}^{N} \left(Y_i T_i - \int_0^T {}_i f'(u) du \right)^2$$

The values Y_i, T_i come from the real-world data. Zero-coupon bonds always have a yield and maturity attached to them. The $\int_0^T (f'(u))^2 du$ term captures the smoothness of the forward curve. If forward rates are highly variable, then market expectations of future interest rates are also highly variable. The $\sum_{i=1}^{N} \left(Y_i T_i - \int_0^{T_i} f'(u) du \right)^2$ is the squared error of our predictions. The parameter α controls the trade-off between smoothness and goodness of fit. Note that if $\alpha = 0$, then the optimal $f(u) \equiv 0$, and if $\alpha \to \infty$ then the optimal f perfectly fits the data.

The actual methods for solving the minimization problem are well beyond our reach, although Filipovic (2009) provides proofs for the existence and uniqueness of the solution. One we have arrived at the optimal $f(u)$, we can compute the yield curve via $Y(t) = \frac{1}{t}\int_0^t f(u)du$.

2.8 Exponential Polynomial Models

The exponential polynomial family contains some of the most well known and used estimation models. Like the smoothing spline, these models estimate the forward curve $f(u)$ with a function $\phi(u|z)$ where z is a set of coefficients. All of the models in the exponential polynomial family has ϕ of the form $p_0 + p_1(u)e^{-\alpha_1 u} + \cdots + p_n(u)e^{-\alpha_n u}$ where p_i is a polynomial of degree

i in the variable u.
Two of the most common models is this family are the Nelson-Siegel and Svensson models.

Nelson-Siegel:

$$\phi(u|z) = z_1 + (z_2 + z_3 u)e^{-z_4 u}$$

Svensson:

$$\phi(u|z) = z_1 + (z_2 + z_3 u)e^{-z_5 u} + z_4 e^{-z_6 u}$$

Again, the actual methods of estimating the forward curve (here this is reduced to solving for the optimal parameters z) are beyond our level. Like in the smoothing spline model, we complete the estimation by backing out the yield curve from the forward curve.

The major advantage of such models is that they are able to accurately capture the term structure model in such few parameters. Since these models are exponential polynomials, smoothness is a built-in feature of the estimates and not something that has to be tuned like the previous models.

2.9 Estimation and Comparison

In this section we will compare the previous spline methods using real world data. Our data comes from the yields of U.S. Treasury bonds. [1]

Maturity	Yield (%)	Maturity	Yield (%)
1 Month	0.01	5 Year	1.72
6 Month	0.09	7 Year	2.41
1 Year	0.13	10 Year	3.00
2 Year	0.39	20 Year	3.68
3 Year	0.76	30 Year	3.92

Table 1: Data from Jan 02 2014, Source: Treasury Department

In Figure 1, we estimate the yield curve according to the previous models. [2] The level of agreement between the models might surprise a reader new to the subject. After all, if the models all accurately predict the same yield curve, why do so many researchers devote their time to improving the models? The answer is that some models have statistically performed better for certain functions and poorly for others. For example, the Svensson model here is known to perform better under nonstandard conditions, like an "inverted yield curve" (here used to describe a yield curve that no longer meets the typical non-decreasing condition). Also, depending on the modeler's needs, she might choose to prioritize goodness of fit over the smoothness of the yield curve. For example, a trader at a major bank will prioritize goodness of fit over smoothness, while a central banker will do the opposite. The BIS (2002) paper further explores the benefits/drawbacks of various models.

[1] Historical data available at treasury.gov under "Daily Treasury Yield Curve Rates"

[2] We rely on finance packages in Python and R for the computation. From Scipy.Interpolate we use UnivariateSpline and Bspline for the cubic-spline and smoothing spline. We use the "YieldCurve" package in R for the Svensson model. Code for the estimation is attached as an appendix.

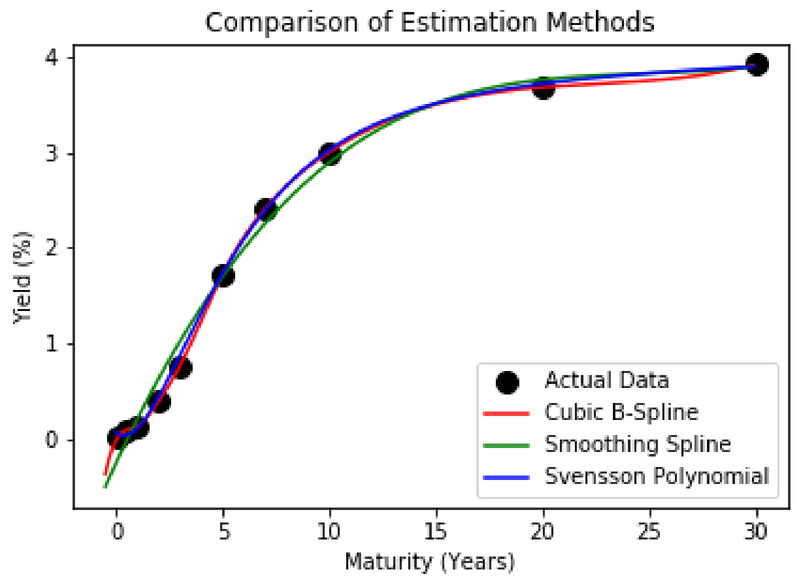

Figure 1: Comparison of Term Structure Estimation Methods

3 Term Structure Models

We now discuss the theoretical underpinnings of term structure models. While the estimation methods introduced in the previous section allow us to estimate the yield curve, the stochastic models in this section provide deeper insight into the relationship between yields and maturities.

3.1 Interest Rate Models

The simplest models have the risk-free interest rate r a constant. However, a more realistic assumption is to model interest rates has stochastic processes. We begin with the one factor stochastic differential equation:

$$dR(t) = \beta(t, R(t))dt + \gamma(t, R(t))d\tilde{W}(t) \tag{1}$$

following the usual notion where $\tilde{W(t)}$ is Brownian Motion under the risk-neutral measure. The discount process $D(t)$ dictates how market participants discount future income relative to current income (i.e. net present value in finance terms).

$$D(t) = e^{-\int_0^t R(s)ds} \tag{2}$$

Definition 3.1. *Zero-Coupon Bond: Contract promising to pay a certain "face value" (usually 1) at a fixed maturity date T.*

The value of such a bond at time t is given by $D(t)B(t,T) = \tilde{\mathbb{E}}[D(T)|\mathcal{F}(t)]$ which implies $B(t,T) = \tilde{\mathbb{E}}[e^{-\int_t^T R(s)ds}|\mathcal{F}(t)]$. Note that this formula satisfies the terminal condition $B(T,T) = 1$.

Theorem 3.1. *Discounted bond prices of a zero-coupon bond are martingales that satisfy the differential equation*

$$f_t(t,r) + (t,r)f_r(t,r) + \frac{1}{2}\gamma^2(t,r)f_{rr}(t,r) - rf(t,r) = 0 \tag{3}$$

Proof. First, to show that that D(t)B(t,T) is a martingale. Under the risk neutral measure $B(t,T) = \tilde{\mathbb{E}}[e^{-\int_t^T R(s)ds}|\mathcal{F}(t)]$ is Brownian Motion by Theorem 5.2.3 in Shreve (2008). Thus D(t)B(t,T) is a martingale.

Second, now that D(t)B(t,T) is a martingale, the Martingale Representation Theorem (MRT) implies that there exists an adapted process $\Gamma(u)$ $0 \le u \le T$ st. $D(t)B(t,T) = B(0,T) + \int_0^t \Gamma(u)dW(u)$.

Writing out the stochastic differential of $D(t)B(t,T)$,

$$\begin{aligned} d(D(t)B(t,T)) =& d(D(t)f(t,R(t)) = f(t,R(t))dD(t) + D(t)df(t,R(t)) \\ =& D(t)[-Rfdt + f_t dt + f_r dR + \frac{1}{2}f_{rr}dt] \\ =& D(t)\underbrace{[-Rfdt + f_t + \beta f_r + \frac{1}{2}\gamma^2 f_{rr}]}_{=0 \text{ by MRT}} dt + D(t)\gamma f_r d\tilde{W} \end{aligned}$$

Thus, the bond prices of a zero-coupon bond must satisfy the stochastic differential equation given in equation 3. □

3.2 Affine Yield Models

Affine yield models are defined by the feature that yields for zero-coupon bond prices are an affine (linear plus constant) function of the interest rate. The prominent models that we will discuss here are the Hull-White and Cox-Ingersoll-Ross (CIR) models.

3.2.1 One-Factor Hull-White Model

In the Hull-White interest rate model, the dynamics of the interest rate is given by:

Definition 3.2.

$$dR(t) = (a - bR(t))dt + \sigma d\tilde{W}(t) \tag{4}$$

Here, $a(t)$, $b(t)$, and $\sigma(t)$ are all deterministic and positive functions of time. The resulting partial differential equation is thus:

$$f_t(t,r) + (a(t) - b(t)r)f_r(t,r) + \frac{1}{2}\sigma^2(t)f_{rr}(t,r) - rf(t,r) = 0 \tag{5}$$

After an initial guess of

$$f_t(t,r) = e^{-rC(t,T)-A(t,T)} \tag{6}$$

to solve the above partial-differential equation, where $C(t,T)$ and $A(t,T)$ are nonrandom, the form is verified and the two functions are given by:

$$C(t,T) = \int_t^T e^{-\int_t^s b(v)dv} ds \tag{7}$$

$$A(t,T) = \int_t^T (a(s)C(s,T) - \frac{1}{2}\sigma^2(s)C^2(s,T))ds \tag{8}$$

As a result, the resulting for the price $B(t,T)$ of a zero-coupon bond is given by:

$$B(t,T) = e^{-R(t)C(t,T)-A(t,T)} \tag{9}$$

where $0 \leq t \leq T$.

3.2.2 One-Factor CIR Model

The one-factor CIR model differs from the Hull-White model with the following form:

Definition 3.3. *One-Factor CIR Model:*

$$dR(t) = (a - bR(t))dt + \sigma\sqrt{R(t)}d\tilde{W}(t) \tag{10}$$

where a, b, and σ are positive and constant. A nice feature of these models is the easy interpretation of the parameters as well as the mean reversion of the interest rate towards a long-run value. The rate of mean reversion is 1/b and the long-run risk-free interest rate is $a * b$, and σ^2 represents the volatility of the risk-free interest rate. The partial differential equation that follows is:

$$f_t(t,r) + (a(t) - br)f_r(t,r) + \frac{1}{2}\sigma^2 r f_{rr}(t,r) - rf(t,r) = 0 \tag{11}$$

Similar to the Hull-White model above, we guess a solution of the form

$$f_t(t,r) = e^{-rC(t,T)-A(t,T)} \tag{12}$$

to solve the above partial-differential equation, where $C(t,T)$ and $A(t,T)$ are again nonrandom. The form is verified and the two functions are given by:

$$C(t,T) = \frac{\sinh(\gamma(T-t))}{\gamma\cosh(\gamma(T-t)) + \frac{1}{2}b\sinh(\gamma(T-t))} \tag{13}$$

$$A(t,T) = \frac{-2a}{\sigma^2} log(\frac{\gamma e^{\frac{1}{2}b(T-t)}}{\gamma\cosh(\gamma(T-t)) + \frac{1}{2}b\sinh(\gamma(T-t))}) \tag{14}$$

Here, $\gamma = \frac{1}{2}\sqrt{b^2 + 2\sigma^2}$. Note that these are solutions to the terminal conditions $A(T,T) = C(T,T) = 0$.

3.2.3 Two-Factor CIR Model

The models discussed so far have thus been one-factor models in that the interest rate is determined by only one stochastic differential equation. As the name suggests, the two-factor CIR model is given by a system of two stochastic differential equations defined by the following:

Definition 3.4.

$$R(t) = \delta_0 + \delta_1 Y_1(t) + \delta_2 Y_2(t) \tag{15}$$

$$dY_1(t) = (\mu_1 - \lambda_{11} Y_1(t) - \lambda_{12} Y_2(t))dt + \sqrt{Y_1(t)}d\tilde{W}_1(t) \tag{16}$$

$$dY_2(t) = (\mu_2 - \lambda_{21} Y_1(t) - \lambda_{22} Y_2(t))dt + \sqrt{Y_2(t)}d\tilde{W}_2(t) \tag{17}$$

where $\delta_0 \geq 0$, $\delta_1 > 0$, $\delta > 0$, $\mu_1 \geq 0$, $\mu_2 \geq 0$, $\lambda_{11} > 0$, $\lambda_{22} > 0$, $\lambda_{12} \leq 0$, and $\lambda_{21} \leq 0$.

To solve the above, we guess a solution of the form

$$f(t, y_1, y_2) = e^{-y_1 C_1(T-t) - y_2 C_2(T-t) - A(T-t)} \tag{18}$$

From the terminal conditions that $C_1(0) = C_2(0) = A(0) = 0$, the following system of ordinary differential equations results:

$$C_1'(\tau) = -\lambda_{11} C_1(\tau) - \lambda_{21} C_2(\tau) - \frac{1}{2} C_1^2(\tau) + \delta_1 \tag{19}$$

$$C_2'(\tau) = -\lambda_{12} C_1(\tau) - \lambda_{22} C_2(\tau) - \frac{1}{2} C_2^2(\tau) + \delta_2 \tag{20}$$

$$A(t, T) = \mu_1 C_1(\tau) + \mu_2 C_2(\tau) + \delta_0 \tag{21}$$

A numerical solution can then be used to solve this system.

4 Conclusion

In this paper we have examined both the estimation methods and theory behind the term structure. We show that the cubic B-spline, smoothing spline, and Svensson methods all provide reliable and consistent estimates of the yield curve, and then we turn the stochastic interest rate models such as the Hull-White and CIR models to better understand the relationship between yields and maturities.

References

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Bolder, D. J. (2001). Affine Term-Structure Models: Theory and Implementation. *Bank of Canada Working Paper Series*.

Brandt, M. (2001). Affine Term Structure Models.

Filipovic, D. (2009). *Springer Finance*.

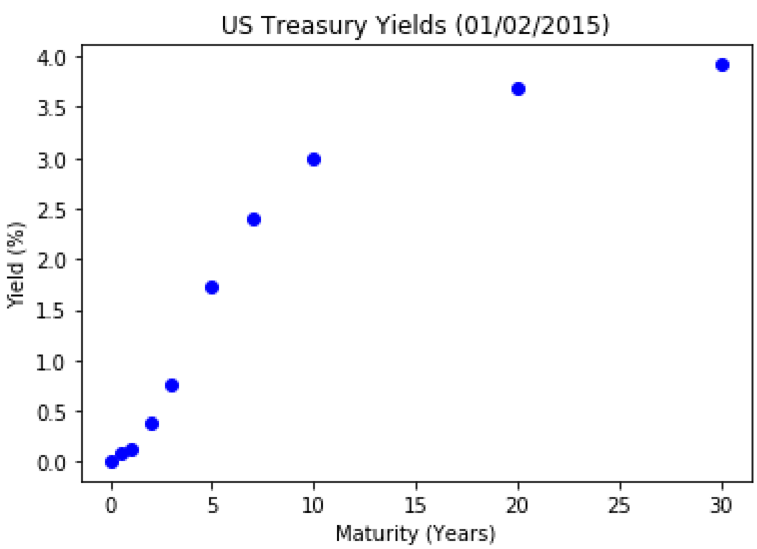
Russell, S. (1992). Understanding the Term Structure of Interest Rates : The Expectations Theory. *St. Louis Fed Publications*.

Shreve, S. (2008). *Springer Finance*.

In [75]:
```
import os
os.chdir('C:\\Users\\WHL\\Documents\\Math 530')
import matplotlib.pyplot as plt
import pandas as pd
import numpy as np
import sys
import math
import QuantLib as ql
from scipy.interpolate import CubicSpline
from scipy.interpolate import BSpline
from scipy.interpolate import UnivariateSpline
from scipy.interpolate import splrep
```

In [4]:
```
#Does the plot of the real data
y = np.array([.01, .09, .13, .39, 0.76, 1.72, 2.41, 3, 3.68, 3.92])
mat = np.array([1/12 , 1/2 , 1, 2, 3, 5 , 7 , 10 , 20 ,30])
```

In [5]:
```
plt.plot(mat, y, 'bo')
plt.ylabel('Yield (%)')
plt.xlabel('Maturity (Years)')
plt.title('US Treasury Yields (01/02/2015)')
plt.savefig('RealData.jpeg')
```



In [6]:
```
#Cubic Spline
cs = CubicSpline(mat, y)
xs = np.arange(-.5, 30, .1)
cube_spline = cs(xs)
```

In [69]:
```
#Cubic B-Spline
alpha = len(mat) - math.sqrt(2*len(mat)) + 1 #on advice on the scipy developer
s
t,c,k = splrep(mat, y, s = 0, k = 3)
cbs = BSpline(t,c,k, extrapolate = True)
cubic_bspline = cbs(xs)
```

In [70]:
```
#Smoothing Spline
ss = UnivariateSpline(mat, y, k = 3, s = 5)
smooth_spline = ss(xs)
```

In [11]:
```
#Svensson Model
sven = pd.read_csv('sven.curve.csv') #Estimates from R package "YieldCurve"
sven['y'] = np.arange(360)/12
```

In [72]:
```
plt.plot(mat, y, 'o',color= 'black', label = 'Actual Data', markersize = 10)
plt.plot(xs, cubic_bspline, label = 'Cubic B-Spline', color = 'red')
plt.plot(xs, smooth_spline, label = 'Smoothing Spline', color = 'green')
plt.plot(sven['y'], sven['x'] , label = 'Svensson Polynomial', color = 'blue')
plt.ylabel('Yield (%)')
plt.legend()
plt.title('Comparison of Estimation Methods')
plt.xlabel('Maturity (Years)')
plt.savefig('Comparison')
```